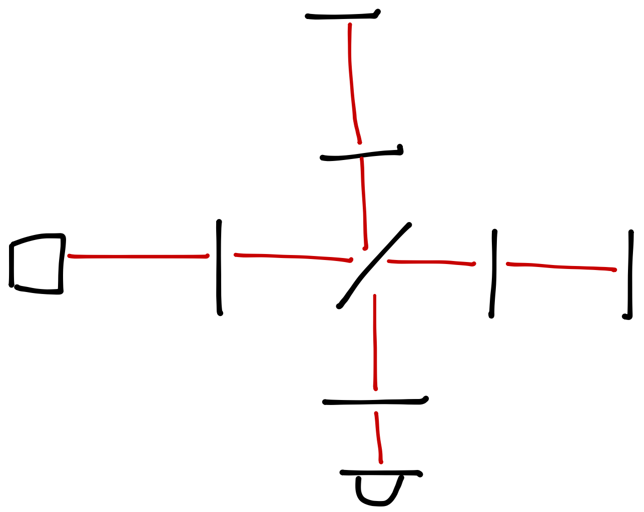
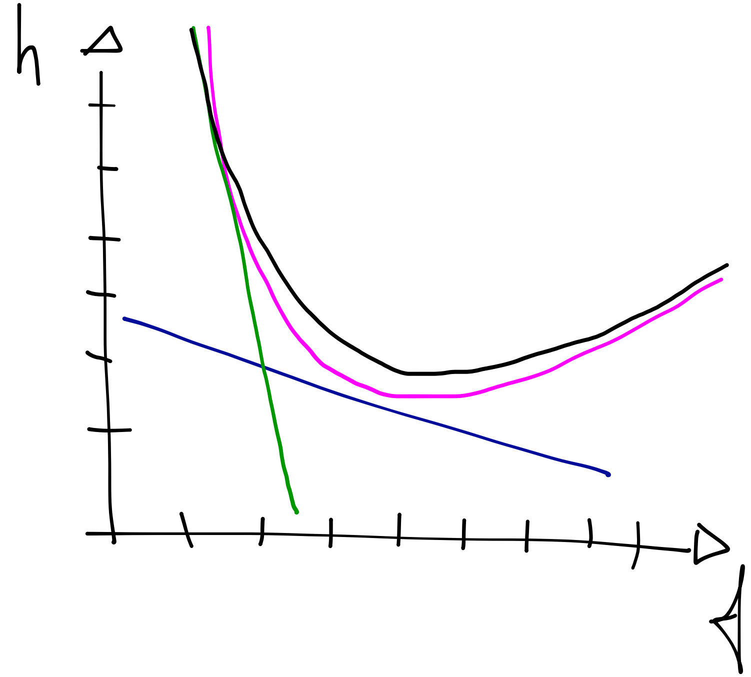
6 MODULATION





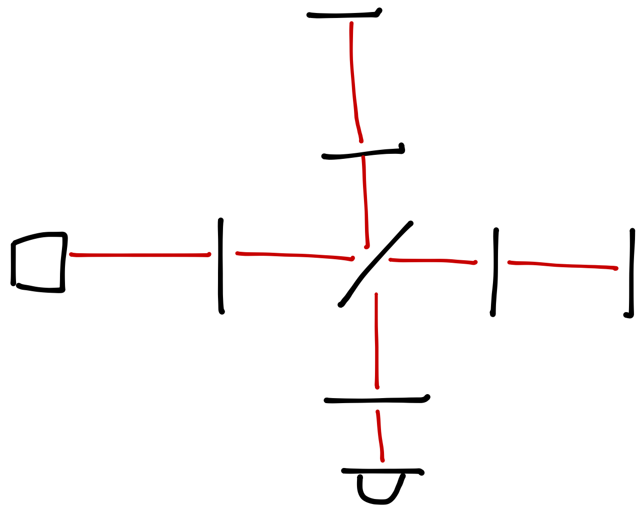
This session:

How to transfer a signal (electric, mechanic) to the beam?

Modulation of light fields!

Compute the transfer function of the interferometer

Full signal propagation



$$\frac{\text{output}(f)}{\text{input}(f)} = \text{response}(f) : \begin{cases} \text{transfer function or} \\ \text{frequency response} \end{cases}$$

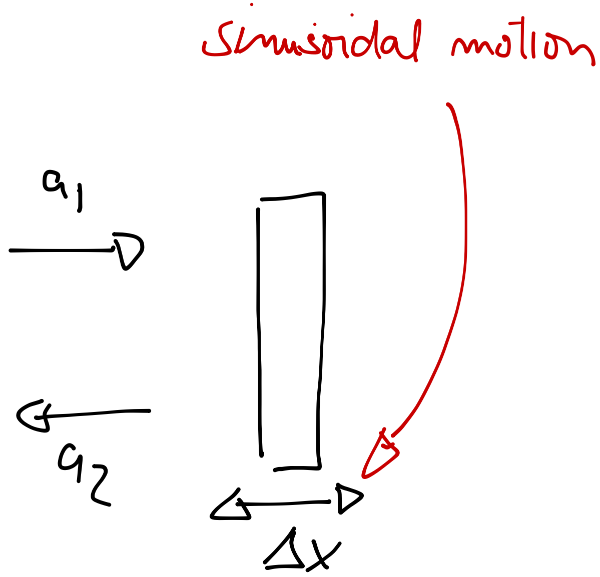
Input: e.g. motion of mirrors

output: signal detected by photo detector in Michelson output.

Transfer of a signal to the light beam

Example: mirror motion

$$\Delta x = m \cdot \cos(\omega_m t)$$



$$a_2 = a_1 \cdot r \cdot e^{-i 2k \Delta x}$$

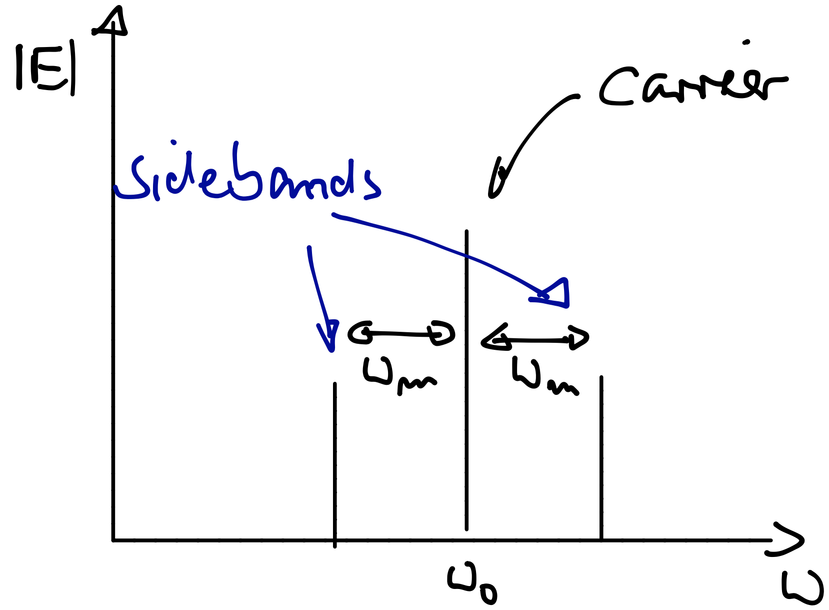
$$= a_1 \cdot r \cdot e^{-i 2km \cos(\omega_m t)}$$

$$|a_2| = r \cdot |a_1| \qquad \varphi_{a_2} = \varphi_{a_1} - 2km \cos(\omega_m t)$$

This is called 'phase modulation'

All optical signals involve some sort of modulation, as before we want to describe/understand this in the frequency domain.

Amplitude modulation



$$E = E_0 e^{i\omega_0 t}\left(1 + m\cos(\omega_m t)\right)$$

$$= E_0 e^{i\omega_0 t}\left(1 + \frac{m}{2} e^{i\omega_m t} + \frac{m}{2} e^{-i\omega_m t}\right)$$

$$= E_0\left(e^{i\omega_0 t} + \frac{m}{2} e^{i(\omega_0+\omega_m)t} + \frac{m}{2} e^{i(\omega_0-\omega_m)t}\right)$$

$$= a_0 e^{i\omega_0 t} + a_+ e^{i\omega_+ t} + a_- e^{i\omega_- t}$$

with $\omega_+ = \omega_0 + \omega_m$
$\omega_- = \omega_0 - \omega_m$

$\Rightarrow$ carrier plus two sidebands

Phase modulation:

$$E = E_0 e^{i(\omega_0 t + \varphi)} = E_0 e^{i(\omega_0 t + m\cos(\omega_m t))} = E_0 e^{i\omega_0 t} e^{im\cos(\omega_m t)}$$

for very small m : $m \ll 1$: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \dots$

$$\Rightarrow e^{im\cos()} = 1 + im\cos()$$

Difference to amplitude modulation!

$$E = E_0 e^{i\omega_0 t}\left(1 + im\cos(\omega_m t)\right)$$

$$= E_0\left(e^{i\omega_0 t} + i\frac{m}{2} e^{i\omega_+ t} + i\frac{m}{2} e^{i\omega_- t}\right)$$

But for large m?

$$e^{im\cos(\omega_m t)} = \sum_{k=-\infty}^{\infty} i^k J_k(m)\, e^{ik\omega_m t}$$

infinite number of sidebands!

Amplitudes? Complicated!

For $m < 1$: $\quad J_k = \frac{1}{k!}\left(\frac{m}{2}\right)^k$

Also $J_{-k} = (-1)^k J_k$

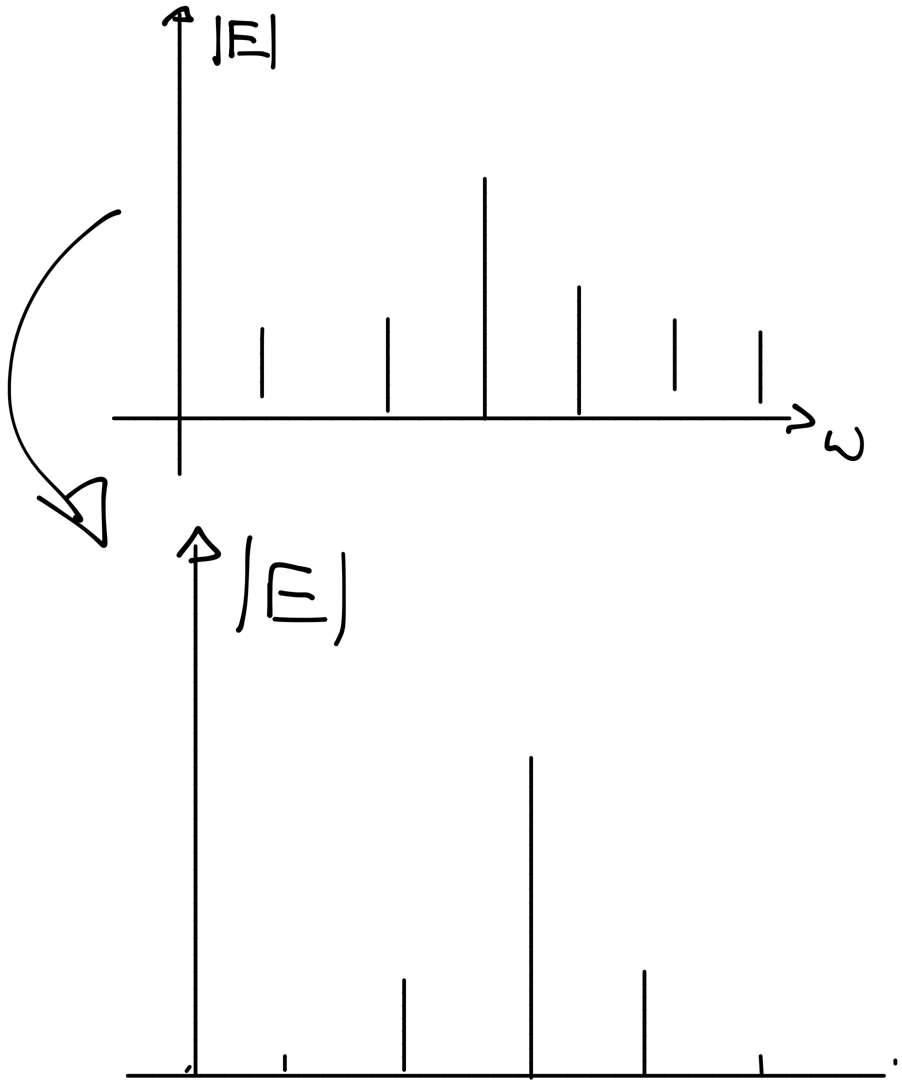
$\Rightarrow J_1 = \frac{m}{2}, \quad J_{-1} = -\frac{m}{2}$

for $|k| < 2$:

$$E = E_0 e^{i\omega_0 t}\left(1 + i\frac{m}{2}e^{i\omega_m t} - \frac{1}{i}\frac{m}{2}e^{-i\omega_m t}\right)$$

$$= E_0\left(e^{i\omega_0 t} + i\frac{m}{2}e^{i\omega_+ t} + i\frac{m}{2}e^{i\omega_- t}\right)$$

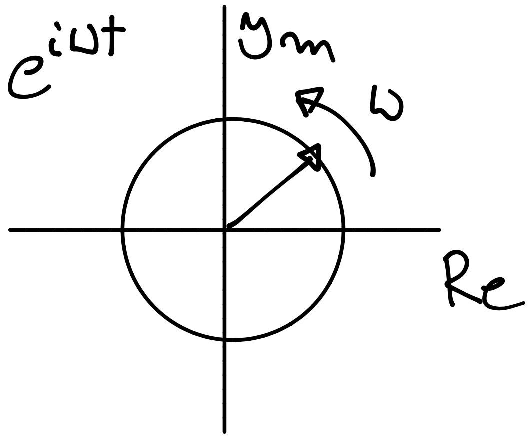
Same as before!



Finding out / remembering which is which:

$$E = E_0\left(e^{i\omega_0 t} + \frac{m}{2} e^{i\omega_+ t} - \frac{m}{2} e^{i\omega_- t}\right)$$

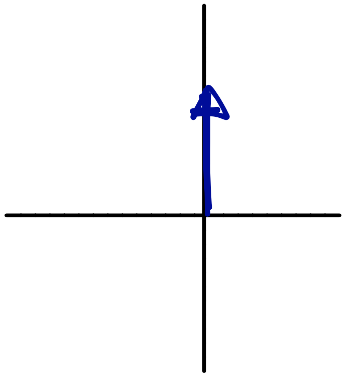
oh!



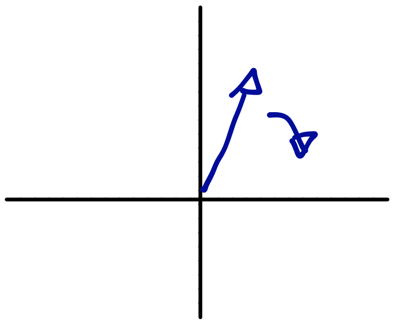
Phasor diagrams: move into rotating frame!

Amp. mod. $E_0\left(e^{i\omega_0 t} + \frac{m}{2} e^{i\omega_+ t} + e^{i\omega_- t}\right)$

carrier

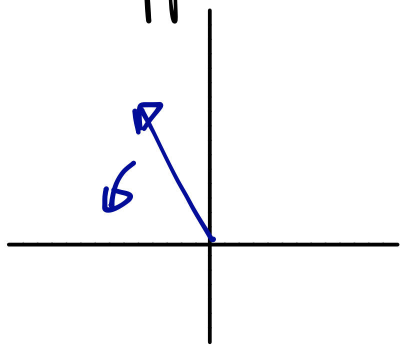


lower SB ω_-



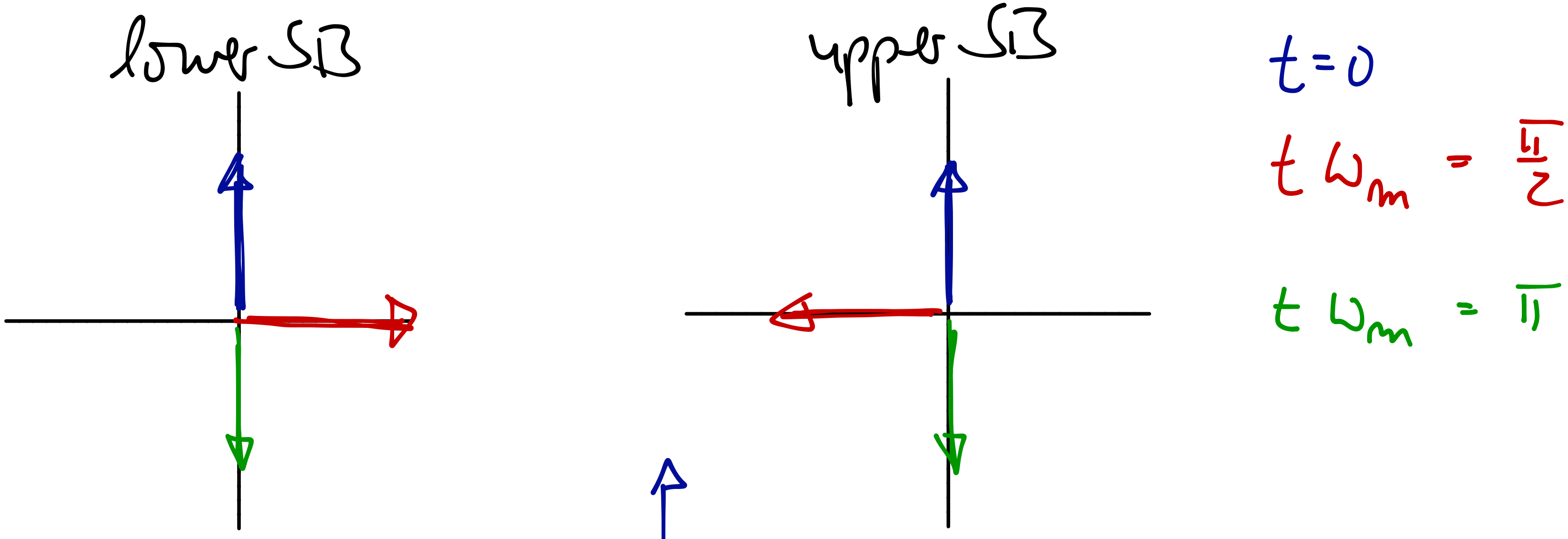
slower than carrier

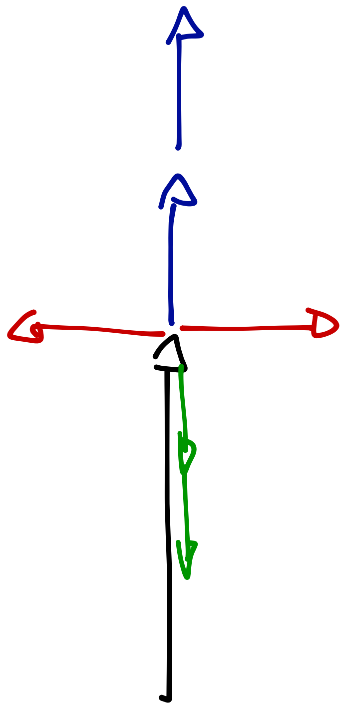
upper SB ω_+



faster than carrier

$$E = E_0 \left(e^{i\omega_0 t} + \frac{m}{2} e^{i\omega_+ t} + \frac{m}{2} e^{i\omega_- t} \right)$$





$\Rightarrow$ Amplitude modulation

$t\omega_m$	up	low
0	1	1
$\pi/2$	i	$-i$
π	-1	-1
$\vdots$		

Amp. mod.

Phase mod.

$t\omega_m$	low	up
0	i	i
$\pi/2$	-1	1
π	$-i$	$-i$
$\vdots$		

$$\text{So}: \quad E = E_0\left(e^{i\omega_0 t} + \frac{m}{2}e^{i\omega_+ t} - \frac{m}{2}e^{i\omega_- t}\right)$$

Phase mod.

Modulation: replace some variable with $\begin{cases} \cos(\omega_m t) \\ (1+\cos(\omega_m t)) \end{cases}$

In the frequency domain this can be described as a number of discret frequency components.

Amp. mod: exactly two sidebands

Phase mod = frequency mod., infinite number of sidebands

convention: phase mod. for small m, often just two sidebands

$$E = E_0 e^{i\omega_0 t}\left(1 + i\frac{m}{2}e^{i\omega_m t} + i\frac{m}{2}e^{-i\omega_m t}\right)$$

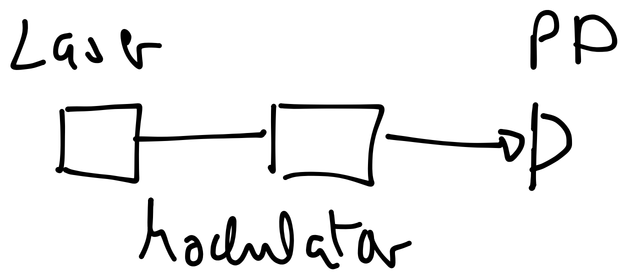
Photodiode $S = EE^*$, $E = a_1 e^{i\omega_0 t} + a_2 e^{i\omega_+ t} + a_2 e^{i\omega_- t}$

$$EE^* = |a_1|^2 + |a_2|^2 + |a_2|^2 + a_1 a_2^* e^{i(\underbrace{\omega_0 - \omega_+}_{=-\omega_m})t} + a_1 a_2^* e^{i(\underbrace{\omega_0 - \omega_-}_{=\omega_m})t} + |a_2|^2 e^{i(\underbrace{\omega_+ - \omega_-}_{2\omega_m})t}$$

$$+ a_1^* a_2 e^{i(\underbrace{\omega_+ - \omega_0}_{=\omega_m})t} + a_1^* a_2 e^{i(\underbrace{\omega_- - \omega_0}_{=-\omega_m})t} + |a_2|^2 e^{i(\underbrace{\omega_- - \omega_+}_{=-2\omega_m})t}$$

$$= |a_1|^2 + 2|a_2|^2 + (a_1^* a_2 + a_1 a_2^*)(e^{i\omega_m t} + e^{-i\omega_m t}) + 2|a_2|^2 \cos(2\omega_m t)$$

$$= |a_1|^2 + 2|a_2|^2 + \mathrm{Re}(a_1 a_2^*) \cos(\omega_m t) \cdot 4 + 2|a_2|^2 \cos(2\omega_m t)$$



Case 1 : Amplitude modulation

$$a_1 = E_0 \quad , \quad a_2 = \frac{m}{2} E_0$$

$$S_{pd} = |E_0|^2 \left(1 + \frac{m^2}{2} + 2m \cos(\omega_m t) + \frac{m^2}{2} \cos(2\omega_m t)\right)$$

Case 2 : phase modulation

$$a_1 = E_0 \quad , \quad a_2 = i \frac{m}{2} E_0$$

$$S_{pd} = |E_0|^2 \left(1 + \frac{m^2}{2} + \frac{m^2}{2} \cos(2\omega_m t)\right)$$

Phase modulation:

- no signal at Ω_m on the photodiode ✓
- power increase $\frac{m^2}{2}$? We used $J_k(m) = \frac{1}{k!}\left(\frac{m}{2}\right)^k + O(m^{k+2})$

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k!\,k!} = 1 - \left(\frac{x}{2}\right)^2 + \cdots$$

$$S_{pd} = \left(1 - \frac{m^2}{4}\right)^2 + \frac{m^2}{2} = 1 - \frac{m^2}{2} + \frac{m^4}{16} + \frac{m^2}{2} = 1 + \frac{m^4}{16} \quad . \text{ Aha!}$$

No signal because $a_1 a_2^* + a_1^* a_2 = 0$

For non-symmetric sidebands: $a_1 a_2^* + a_1^* a_3 \neq 0$

For symmetric sidebands but $a_1 a_2^*$ not imaginary $a_1 a_2^* + a_1^* a_2 \neq 0$

In other words phase difference between a_1 and a_2 not 90 deg

Or: if $a_2 = i$, we get a signal whenever the phase of the light of the carrier is not 0 → <u>phase sensitive signal!</u>

Michelson and differential mode

One arm: $E_1 = \sqrt{\tfrac{1}{2}}\, E_0\, e^{i\omega_0 t}\left(1 + i\frac{m}{2} e^{i\omega_m t} + i\frac{m}{2} e^{-i\omega_m t}\right)$

Other arm: $E_2 = i\sqrt{\tfrac{1}{2}}\, E_0\, e^{i\omega_0 t}\left(1 - i\frac{m}{2} e^{i\omega_m t} - i\frac{m}{2} e^{-i\omega_m t}\right)$

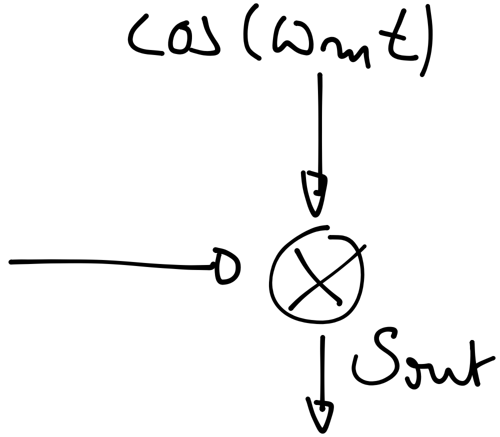
Output $E_{out} = \sqrt{\tfrac{1}{2}} \cdot \left(i E_1 - E_2\right) = \frac{m}{2} E_0 \left(e^{i\omega_m t} + e^{-i\omega_m t}\right) e^{i\omega_0 t}$

Need to at a field at ω_0, called local oscillator, eg. $E_0 e^{i\omega_0 t}$

Then this beomes amplitude modulation at ω_m.

Mixer + Lowpass:

$$S = S_0 + S_1\cos(\omega_m t) + S_2\cos(2\omega_m t)$$



$$S_{out} \sim S\cdot\cos(\omega_m t) = S_0\cos() + S_1\cos^2() + S_2\cos()\cdot\cos(2\omega_m t)$$

$$\cos^2() \to \frac{1}{2}\left(1 + \cos(2\omega_m t)\right)$$

$$\cos()\cdot\cos(2\omega_m t) \to \frac{1}{2}\cos((\omega_m - 2\omega_m)t) + \frac{1}{2}\cos((\omega_m + 2\omega_m)t)$$
$$= \frac{1}{2}\left(\cos(\omega_m t) + \cos(3\omega_m t)\right)$$

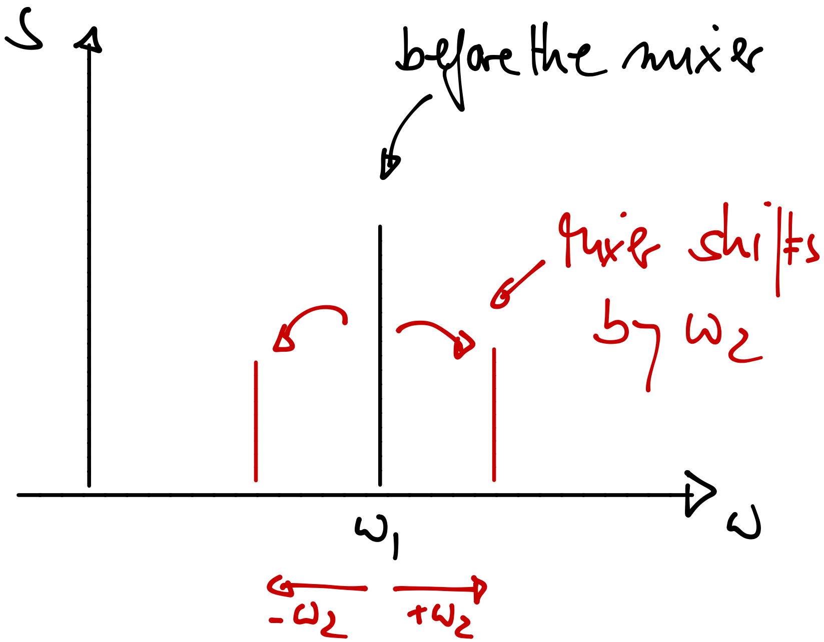
DC $\to \omega_m$

$\omega_m \to$ DC $+ 2\omega_m$

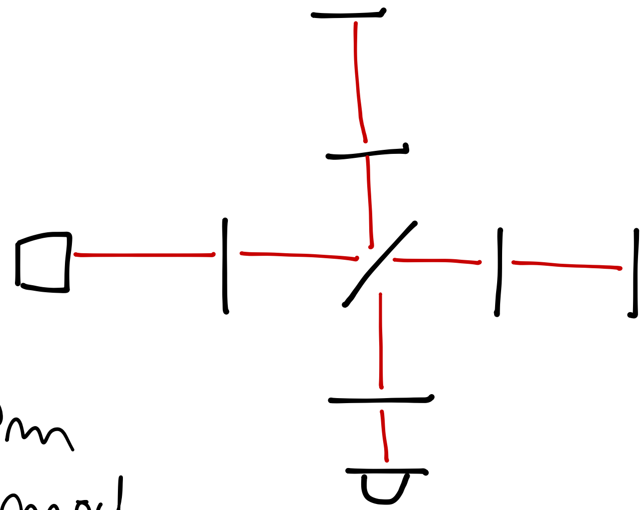
$2\omega_m \to \omega_m + 3\omega_m$

$S_{out} \to \boxed{\text{Lowpass}} \to S_{LP}$

$S_{LP} = \frac{1}{2}\cdot S_1$ ✓



Full signal propagation



- mirror motion at ω_m
- phase modulation of light
- two new components of E field at $\omega_0 \pm \omega_m$
- Michelson turns phase mod. to amplitude mod
- photodiode + mixer + low pass
- output : amplitude of signal at ω_m
- proportional to initial amplitude * interferometer response

$$\frac{\text{output}(f)}{\text{input}(f)} = \text{response}(f) : \begin{cases} \text{transfer function or} \\ \text{frequency response} \end{cases}$$

Summary:

- compute the transfer function of the detector from mirror motion to photo diode output
- signal causes light modulation
- modulation causes new field components at $\omega_0 \pm \omega_m$
- photodiode + mixer + lowpass gives amplitude at component at ω_m